Bell theorem involving all possible local measurements

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(August 3, 1999)

Abstract

The Bell theorem for a pair of two-state systems in a singlet state is formulated for the entire range of measurement settings.

Typeset using REVTEX

The Bell theorem is usually formulated with the help of the Clauser-Horne [1] or the CHSH inequality [2]. These inequalities are satisfied by any local realistic theory and are violated by quantum mechanical predictions. They involve two apparatus settings at each of the two sides of the experiment. However, generalisation to more than two settings at each side are possible [3], [4], [5], [6].

There are several motivations for such generalisations. First of all new Bell inequalities may be more appropriate in some experimental situations, e.g., the chained Bell inequalities can reveal violation of local realism for the Franson type experiment [7]. Also, the academic question, why only two settings at each side, is that always necessary, is interesting in itself. Further, many of the currently performed quantum interferometric Bell tests did not involve stabilisation of the interferometers at specified settings optimal for the standard Bell inequalities, but rather involved sample scans of the entire interferometric patterns. Thus it is useful to have inequalities that are *directly* applicable to such data.

Here we present a Bell-type inequality that involves all possible settings of the local measuring apparatus for a pair of two-state systems, which is always equivalent to two spin $\frac{1}{2}$ particles. The method applied is a development of the one given in [5]. However, here we do not restrict ourselves to pairs of coplanar settings (in the meaning appropriate for two Stern-Gerlach apparatuses).

Our method has two characteristic traits. The first one is that it indeed involves the entire range of the measurement parameters. By this, e.g., it distinguishes itself from the limits of infinitely many settings at each side of the so-called chained inequalities [4], in which not every *pair* of possible settings is utilised. The second one is that the method involves the quantum prediction from the very beginning. As we shall see the quantum prediction determines the structure of our Bell inequality.

In a standard Bell-type experiment one has a source emitting two particles, each of which propagates towards one of two spatially separated measuring devices. The particles are described by the maximally entangled state, e.g.,

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|+\rangle_1|-\rangle_2 - |-\rangle_1|+\rangle_2), \tag{1}$$

where $|+\rangle_1$ is the state of the first particle with its spin directed along the versor $\vec{z}$ of a certain frame of reference ($-$ denotes the opposite direction), etc.

Let as assume that every measuring device is a Stern-Gerlach apparatus, which measures the observable $\vec{n}\cdot\vec{\sigma}$, where $n = a, b$ (a for the first observer, b for the second one), $\vec{n}$ is a unit vector representing direction at which observer n makes a measurement and $\vec{\sigma}$ is a vector the components of which are standard Pauli matrices. The family of observables $\vec{n}\cdot\vec{\sigma}$ covers all possible dichotomic observables for a spin $\frac{1}{2}$ system, endowed with a spectrum consisting of ± 1.

In each run of the experiment every observer obtains one of the two possible results of measurement, ± 1. The probability of obtaining by the observer a the result $m = \pm 1$, when measuring the projection of the spin of the incoming particle at the direction $\vec{a}$, and the result $m' = \pm 1$ by the observer b, when measuring the projection of spin of the incoming particle at the direction $\vec{b}$ is equal to

$$P_{QM}(m, m'; \vec{a}, \vec{b}) = \tfrac{1}{4}(1 - mm'\vec{a}\cdot\vec{b}). \tag{2}$$

In a real experiment, however, one cannot expect that the observed probabilities will follow (2). Therefore, we will allow that the interference pattern is of a reduced visibility. In such a case (2) should be replaced by

$$P_{QM}(m, m'; \vec{a}, \vec{b}) = \tfrac{1}{4}(1 - mm'V\vec{a}\cdot\vec{b}), \tag{3}$$

where $0 \leq V \leq 1$ stands for the visibility.

From the perspective of local realism one can try to give a more complete specification of the state of a member of the ensemble of two-particle systems than the one given by $|\Psi\rangle$. The usual approach is to define a space of hidden states Λ and a probability distribution $\rho(\lambda)$ of such states and to represent the probability of specific results by

$$P_{HV}(m, m'; \vec{a}, \vec{b}) = \int_\Lambda d\lambda \rho(\lambda) P_A(m|\vec{a}, \lambda) P_B(m'|\vec{b}, \lambda), \tag{4}$$

where $P_A(m|\vec{a},\lambda)$ is the probability that for given λ and for the local observable defined by the parameter $\vec{a}$ the first observer obtains as a result the value m ($P_B(m',\vec{b},\lambda)$ plays the same role on the other side of the experiment).

We want to check if it is possible to recover quantum mechanical probabilities P_{QM} using probabilities P_{HV} based on the assumptions of local realistic theories. We will follow the reasoning first given in [5] which is based on the following simple geometric observation. Assume that one knows the components of a certain vector q (the *known* vector) belonging to some Hilbert space, whereas about a second vector h (the *test* vector) one is only able to establish that its scalar product with q satisfies the inequality $\langle h|q\rangle < ||q||^2$. The immediate implication is that these two vectors cannot be equal $q \neq h$.

To apply the above simple geometric fact to our case we must define appropriate Hilbert space. Because we deal with functions $P_{QM}(m,m';\theta_a,\phi_a,\theta_b,\phi_b)$ and $P_{HV}(m,m';\theta_a,\phi_a,\theta_b,\phi_b)$ that depend on discrete numbers m,m' and continuous variables θ_n,ϕ_n, where $\vec{n} = (\sin\theta_n\cos\phi_n, \sin\theta_n\sin\phi_n, \cos\theta_n)$ it is convenient to define the scalar product of certain two real functions f and g as

$$\langle f|g\rangle = \sum_{m=-1}^{1}\sum_{m'=-1}^{1}\int d\Omega_a \times \int d\Omega_b f(m,m';\theta_a,\phi_a,\theta_b,\phi_b)g(m,m';\theta_a,\phi_a,\theta_b,\phi_b), \tag{5}$$

where $d\Omega_n = \sin\theta_n d\theta_n d\phi_n$ is the rotationally invariant measure on the sphere of radius one. Our *known* vector is P_{QM}, whereas the *test* one is P_{HV}.

One has

$$||P_{QM}||^2 = \langle P_{QM}|P_{QM}\rangle = (2\pi)^2 + V^2\frac{4\pi^2}{3}. \tag{6}$$

To estimate the scalar product $\langle P_{QM}|P_{HV}\rangle$ one has to use the specific structure of probabilities that are described by local hidden variables (LHV) (4). Since P_{HV} is a weighted average over the hidden parameters one can make the following estimation

$$\langle P_{QM}, P_{HV}\rangle \leq \max_{\lambda\in\Lambda}\left[\sum_{m,m'=-1}^{1}\int d\Omega_a \int \Omega_b P_A(m|\vec{a},\lambda)\right.$$
$$\left.\times P_B(m'|\vec{b},\lambda)\frac{1}{4}(1-mm'V\vec{a}\cdot\vec{b})\right]. \tag{7}$$

Since $\sum_{m=-1}^{1} P_A(m|\vec{a},\lambda) = \sum_{m'=-1}^{1} P_B(m'|\vec{b},\lambda) = 1$, the first term of (7) satisfies

$$\frac{1}{4}\sum_{m,m'=-1}\int d\Omega_a \int d\Omega_b P_A(m|\vec{a},\lambda)P_B(m'|\vec{b},\lambda)$$
$$= (2\pi)^2. \tag{8}$$

We transform the other term of (7) to a more convenient form

$$\frac{1}{4}\sum_{m,m'=-1}^{1}\int d\Omega_a \int d\Omega_b mm' P(m|\vec{a},\lambda)P(m'|\vec{b},\lambda)V\vec{a}\cdot\vec{b}$$
$$= \frac{1}{4}\int d\Omega_a \int d\Omega_b I_a(\vec{a},\lambda)I_b(\vec{b},\lambda)V\vec{a}\cdot\vec{b}, \tag{9}$$

where $I_n(\vec{n},\lambda) = \sum_{m=-1}^{1} mP_n(m|\vec{n},\lambda)$, and one has $|I_n(\vec{n},\lambda)| \leq 1$ $(n = a, b)$.

The scalar product of two three dimensional vectors $\vec{a}$ and $\vec{b}$ that appears in (9) can be written as $\vec{a}\cdot\vec{b} = \sum_{k=1}^{3} a_k(\theta_a,\phi_a)b_k(\theta_b,\phi_b)$, where

$$\vec{n} = (n_1, n_2, n_3)$$
$$= (\sin\theta_n\cos\phi_n, \sin\theta_n\sin\phi_n, \cos\theta_n). \tag{10}$$

Therefore (9) reads

$$\frac{V}{4}\sum_{k=1}^{3}\int d\Omega_a I_a(\theta_a,\phi_a,\lambda)a_k(\theta_a,\phi_a)$$
$$\times\int d\Omega_b I_b(\theta_b,\phi_b,\lambda)b_k(\theta_b,\phi_b). \tag{11}$$

We notice here that our expression is a sum of three terms, each of which is a product of two integrals.

The functions in (11) are square integrable, i.e. integrals $\int d\Omega_n |I_n(\theta_n,\phi_n,\lambda)|^2$ and $\int d\Omega_n |n_k(\theta_n,\phi_n)|^2$ exist (we remind that $|I_n(\theta_n,\phi_n,\lambda)| \leq 1$ which guarantees the existence of the first integral). This all allows us to use formalism of Hilbert space of square integrable functions on the unit sphere, which we denote as $L^2(S^3)$.

The functions $n_k(\theta_n, \phi_n)$ fulfil the orthogonality relation $\int d\Omega_n n_k(\theta_n, \phi_n) n_l(\theta_n, \phi_n) = \frac{4\pi}{3}\delta_{kl}$. Thus, if we normalise n_k (i.e. we divide them by their norm, which is $\sqrt{\frac{4\pi}{3}}$) we can interpret the integral $\alpha_k^n(\lambda) = \sqrt{\frac{3}{4\pi}} \int d\Omega_k I_n(\theta_n, \phi_n, \lambda) n_k(\theta_n, \phi_n)$ as a k-th coefficient of the *projection* of $I_n(\theta_n, \phi_n, \lambda)$ into a three dimensional subspace of $L^2(S^3)$ spanned by the (normalised) basis functions $\sqrt{\frac{3}{4\pi}} n_k(\theta_n, \phi_n)$ $(k = 1, 2, 3)$. For later reference we will call this space $\Sigma(3)$. Therefore (9) transforms into

$$V \frac{\pi}{3} \sum_{k=1}^{3} \alpha_k^a(\lambda) \alpha_k^b(\lambda). \tag{12}$$

Denoting the projection of $I_n(\theta_n, \phi_n, \lambda)$ into $\Sigma(3)$ by $I_n^{||}(\theta_n, \phi_n, \lambda)$ and using the Schwartz inequality we arrive at

$$\frac{\pi}{3} \sum_{k=1}^{3} \alpha_k^a(\lambda) \alpha_k^a(\lambda) \leq \frac{\pi}{3} ||I_a^{||}(\cdot, \lambda)|| ||I_b^{||}(\cdot, \lambda)||. \tag{13}$$

Therefore, our last step is to calculate the maximal possible value of the norm $||I_n^{||}(\cdot, \lambda)||$. Since the length (norm) of a projection of a vector into a certain subspace is equal to the maximal value of its scalar product with any normalised vector belonging to this subspace, the norm $||I_n^{||}(\cdot, \lambda)||$ is given by

$$||I_n^{||}(\cdot, \lambda)|| = \max_{|\vec{c}|=1} [\sqrt{\tfrac{3}{4\pi}} \int d\Omega_n I_n(\theta_n, \phi_n, \lambda) \textstyle\sum_{k=1}^{3} c_k n_k(\theta_n, \phi_n), \tag{14}$$

where $\vec{c} = (c_1, c_2, c_3)$ and $|\vec{c}| = \sum_{k=1}^{3} c_k^2 = 1$. Because $|I_n(\vec{a}, \lambda)| \leq 1$ one has

$$||I_n^{||}(\cdot, \lambda)|| \leq \max_{|\vec{c}|=1} [\sqrt{\tfrac{3}{4\pi}} \int d\Omega_n | \textstyle\sum_{k=1}^{3} c_k n_k(\theta_n, \phi_n)|]. \tag{15}$$

Every vector $\vec{c}$ can be obtained by a certain rotation of the versor $\vec{z}$. Such a rotation is represented by an orthogonal matrix $\hat{O}$ belonging to the rotation group $SO(3)$. Therefore, (15) can be rewritten as

$$||I_n^{||}(\cdot, \lambda)|| \leq \max_{\hat{O}} [\sqrt{\tfrac{3}{4\pi}} \int d\Omega_n |\hat{O}\vec{z} \cdot \vec{n}(\theta_n, \phi_n)|], \tag{16}$$

where the maximum is taken over all possible rotation matrices $\hat{O}$. Since $|O\hat{z} \cdot \vec{n}(\theta_n, \phi_n)|$ is the modulus of the scalar product of two ordinary three dimensional vectors, it is equal to

$|\vec{z}\cdot\hat{O}^{-1}\vec{n}(\theta_n,\phi_n)|$. An active rotation of the vector $\vec{n}$ is equivalent to a (passive) change of the spherical coordinates. Utilising the fact that the measure $d\Omega_n$ is rotationally invariant we see that

$$||I_n^{||}|| \leq \int d\Omega_n |\sqrt{\frac{3}{4\pi}}\cos\theta_n| = 2\pi\sqrt{\frac{3}{4\pi}}. \tag{17}$$

Therefore (13) is not greater then $\frac{1}{4}(2\pi)^2$, which with (6) and (8) gives us the following inequalities

$$||P^{QM}||^2 = (2\pi)^2 + \frac{V^2}{3}(2\pi)^2 > (2\pi)^2 + \frac{V}{4}(2\pi)^2 \geq \langle P^{QM}, P^{HV}\rangle. \tag{18}$$

This inequality is violated by quantum predictions provided that the visibility V is higher then 75%. Please notice that the right hand inequality is a form of a "functional" Bell inequality. It simply gives the upper bound for the value of a certain functional defined on the local realistic probability functions P_{HV}. The left hand inequality shows that the insertion of P_{QM} into the functional Bell inequality leads to its violation provided $V > 0.75$. The characteristic trait of our functional Bell inequality is that its form is defined by the quantum prediction P_{QM}.

The threshold visibility for two particle interference to violate the inequality (18) is lower than in the case of coplanar settings [5], for which the critical visibility is $\frac{8}{\pi^2}$. Also, it is lower than the one given recently by Gisin [6]. For his inequalities involving arbitrary many settings the threshold visibility equals $V = \frac{\pi}{4}$. The chained inequalities, for evenly spaced settings, with the number of settings going to infinity, have the property that the critical visibility approaches in the limit 1.

The question of the threshold visibility gained recently new importance. Two-particle interferometry has been recently extended to interference of photons which originate from independent sources [8]. Thus far the visibility is much lower than in the standard Bell-type tests [9]. Therefore every percentage point chopped off the maximal visibility for the two-particle fringes that may still hide of local and realistic model seems to be of importance. The method presented here can easily be adapted to cases of finite number of local parameter

settings (compare e.g. [5]). Therefore it can be applied directly to experimental data (which involve sequences of numbers, rather than continuous functions).

Acknowledgements MZ was supported by the University of Gdansk Grant No BW/5400-5-0264-9. DK was supported by the KBN Grant 2 P03B 096 15.

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